Ideal and anti-ideal decision making units: A fuzzy DEA approach

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Abstract

In this paper, by introducing two virtual decision-making units (DMUs) called ideal DMU (IDMU) and anti-ideal DMU (ADMU) with fuzzy inputs-outputs, the efficiency evaluation of DMUs are done by fuzzy data envelopment analysis (FDEA). Therefore, we evaluate DMUs from the perspective of the best and worst possible relative efficiency. For each DMU two efficiencies are calculated while inputs and outputs are fuzzy. These two distinctive efficiencies are combined with the closeness coefficient (CC) index. The CC index is then used for an overall ranking of all DMUs. Finally, we compare the result of proposed fuzzy DEA model with León et al.’s (2003) results by representing a numerical example.

Keywords: DEA; Fuzzy mathematical programming; Ideal and anti-ideal decision making unit; Ranking

1. Introduction

Data envelopment analysis (DEA) is a mathematical optimization technique that measures the relative efficiency of decision making units (DMUs) with multiple input–output. Charnes et al. (1978) first proposed DEA as an evaluation tool to measure and compare a DMU’s relative efficiency. Their model which is commonly referred to as CCR model, assumed constant returns to scale. It was developed for variable returns to scale by Banker et al. (1984). That is commonly referred to as BCC model (Cooper et al., 2000).

Evaluating the performance of activities or organizations by conventional DEA models requires crisp input/output data. However, in real-world problems inputs and outputs are often imprecise. The DEA models with fuzzy data can more realistically represent real-world problems than the traditional DEA models. Fuzzy set theory allows linguistic data to be used straightly within the DEA models. Fuzzy DEA (FDEA) models take the form of fuzzy linear programming models.

We can find several fuzzy approaches to the assessment of efficiency in the DEA literature. Sen-gupta (1992) proposed a fuzzy mathematical programming approach in which fuzziness was incorporated into DEA model by defining tolerance levels on both objective function and constraint violations. Triantis and Girod (1998) suggested a mathematical programming approach through transforming fuzzy input-output data into crisp data using membership function values. Efficiency scores were computed for different values of membership functions and then averaged. Kao and Liu (2000; 2003) introduced a technique which transforms a fuzzy DEA model to a family of crisp DEA models by applying the \( \alpha \) -level (also called \( \alpha \)-cuts) approach. Despotis and Smirlis (2002) and Entani et al. (2002) proposed a DEA model with an interval efficiency consisting of efficiencies obtained from the pessimistic and the optimistic viewpoints. Their models deal with fuzzy data. Lertworasirikul et al. (2003) proposed a possibility approach in which
fuzzy DEA model was transformed into possibility DEA model by using possibility measures on fuzzy events. Hougaard’s (2005) approach allows the decision makers to use scores of technical efficiency in combination with other sources of information as expert opinions for instance. The \( \alpha \)-cut approach was also adopted by Saati et al. (2002), who defined the fuzzy CCR model as a possibilistic-programming problem and transformed it into an interval programming using the concept of \( \alpha \)-cuts. Guo and Tanaka (2001) proposed a fuzzy CCR model in which fuzzy constraints including fuzzy equalities and fuzzy inequalities were all converted into crisp constraints by predefining a possibility level and using the comparison rule for fuzzy numbers. Recently, Hatami-Marbini et al. (2009) extended the idea of ranking fuzzy numbers to the fuzzy DEA model, and their method was applied to evaluation bank branches. Hatami-Marbini and Saati (2009) also developed a new method to deal with RTS of efficient DMUs in fuzzy BCC when \( u_0 \) is a fuzzy variable. Here, we are particularly interested in the approach by León et al. (2003), which uses the possibilistic programming. In the present paper we also utilize possibilistic programming techniques to approach the problem of the measurement of efficiency. Some interests are obtained with respect to both computational and interpretative aspects.

In this paper, CCR model is extended to be a fuzzy DEA model and introduced virtual ideal DMU (IDMU) and anti-ideal DMU (ADMU) in DEA model. Section 3 presents proposed approach for solving fuzzy DEA models. Subsequently, we consider a numerical example provided by León et al. (2003) to validate our method in Section 4. The paper is concluded in Section 5.

2. Preliminary definitions

In this section we are simply recalling how to perform the basic operations of arithmetic of fuzzy numbers and introducing IDMU and ADMU in DEA model (Zimmermann, 2005; Klir and Yuan, 1995; Wang and Luo, 2006). They are expressed as follows.

Definition 1. Fuzzy number \( \tilde{M} \) is said to be an LR-fuzzy number, \( \tilde{M} = (p_L, q_R, \alpha_L, \beta_L)^{L,R} \) if and only if its membership function has the following form:

\[
\mu_{\tilde{M}}(x) = \begin{cases} 
L(p_L - x) & x \leq p_L \\
1 & p_L \leq x \leq q_R \\
R(q_R - x) & x \geq q_R
\end{cases} \tag{1}
\]

where \( L \) and \( R \) are reference functions, i.e., \( L; R: [0, \infty) \rightarrow [0, 1] \) are strictly decreasing in \( supp(\tilde{M}) = \{x : \mu_{\tilde{M}}(x) > 0\} \) and upper semi-continuous functions such that \( L(0) = R(0) = 1 \). If \( supp(\tilde{M}) \) is a bounded set, \( L \) and \( R \) are defined on \([0,1]\) and satisfy \( L(1) = R(1) = 0 \).

In fuzzy linear programming (FLP), the min \( T \)-norm is the most applied to evaluate a linear combination of fuzzy quantities. In particular, for a given set of LR-fuzzy numbers \( \tilde{d}_j = (m^L_{jx}, n^R_{jx}, \alpha^L_j, \beta^R_j)^{L,R} \), \( j = 1, \ldots, n \) and some nonnegative \( x_j \), we have that:

\[
\sum_{j=1}^{n} \tilde{d}_j x_j = \left[ \sum_{j=1}^{n} m^L_{jx} x_j, \sum_{j=1}^{n} n^R_{jx} x_j, \sum_{j=1}^{n} \alpha^L_j x_j, \sum_{j=1}^{n} \beta^R_j x_j \right]^{L,R} \tag{2}
\]

where \( L \) and \( R \) are the common left and right reference functions, and \( \sum_{j=1}^{n} \tilde{d}_j x_j \) denotes the combination \( \tilde{d}_1 x_1 \oplus \tilde{d}_2 x_2 \oplus \ldots \oplus \tilde{d}_n x_n \). Literature review reveals that multitudes of fuzzy number ranking methods exist (Wang and Kerre, 2001(a); Wang and Kerre, 2001(b); Bortolan and Degani, 1985). The ranking process depends heavily on the environment or the framework of a problem. Here, we consider an approach of fuzzy number ranking that introduce as the following two definitions:

Definition 2. Let \( \tilde{M} \) and \( \tilde{N} \) be two fuzzy numbers. Then, membership function \( \mu_{\tilde{M} \cdot \tilde{N}}(x) \) is defined as:

\[
\mu_{\tilde{M} \cdot \tilde{N}}(x) = \sup_{x = x} \{ \mu_{\tilde{M}}(s) \wedge \mu_{\tilde{N}}(t) \} \tag{3}
\]
Based on “fuzzy max” operator, Dubois and Prade (1980) recount the Definition 3.

**Definition 3.** Considering two fuzzy numbers \( \tilde{M} \) and \( \tilde{N} \),
\[
\tilde{M} \geq \tilde{N} \iff \tilde{M} \vee \tilde{N} = \tilde{M}
\] (4)

Tanaka et al. (1984), Leon et al. (2003) and Ramík and Římaňek (1985) have formulated FLP problems by using this order. In fact, Ramík and Římaňek (1985) provided an operative characterization of (3) in terms of the \( \alpha \)-cut.

**Theorem.** Let \( \tilde{M} \) and \( \tilde{N} \) be two fuzzy numbers. Then, \( \tilde{M} \vee \tilde{N} = \tilde{M} \) if and only if, \( \forall \alpha \in [0,1] \) the two statements below hold (Ramík and Římaňek, 1985):
\[
\inf \{ s : \mu_{\tilde{M}}(s) \geq \alpha \} \geq \inf \{ t : \mu_{\tilde{N}}(t) \geq \alpha \},
\sup \{ s : \mu_{\tilde{M}}(s) \geq \alpha \} \geq \sup \{ t : \mu_{\tilde{N}}(t) \geq \alpha \}
\] (5)

In particular, if two LR-fuzzy numbers \( \tilde{M} = (p^L, q^R, \alpha^L, \beta^R)_{L \vee R} \) and \( \tilde{N} = (b^L, h^R, \xi^L, \mu^R)_{L \vee R} \) have supported to (5) and both \( L = L' \) and \( R = R' \), then:
\[
p^L \geq b^L, \quad p^L - \alpha^L \geq b^L - \xi^L, \quad q^R \geq h^R, \quad q^R + \beta^R \geq h^R + \mu^R
\] (6)

**Definition 4.** An IDMU is a virtual DMU that uses the least inputs to generate the most outputs. While an ADMU is a virtual DMU which consumes the most inputs but produces the least outputs.

Let \( x_{ij} (i = 1, \ldots, m, j = 1, \ldots, n) \) and \( y_{rj} (r = 1, \ldots, s, j = 1, \ldots, n) \) are the \( i \)th input and the \( r \)th output of DMU \( j \), respectively. An ideal IDMU and ADMU in DEA are defined as follows (exact data) (Wang and Luo, 2006):

**IDMU** \((x_{ij}^{\min}, y_{rj}^{\max})\)

**ADMU** \((x_{ij}^{\max}, y_{rj}^{\min})\)

where:
\[
x^{\min}_i = \min_j \{x_{ij}\}, \quad x^{\max}_i = \max_j \{x_{ij}\}, \quad \forall i
\]
\[
y^{\min}_r = \min_j \{y_{rj}\}, \quad y^{\max}_r = \max_j \{y_{rj}\}, \quad \forall r
\]

On the other hand, Let \( \tilde{x}_{ij} = (x^{L}_{ij}, x^{R}_{ij}, \alpha^{L}_{ij}, \alpha^{R}_{ij}) \) and \( \tilde{y}_{rj} = (y^{L}_{rj}, y^{R}_{rj}, \beta^{L}_{rj}, \beta^{R}_{rj}) \) are the \( i \)th fuzzy input and the \( r \)th fuzzy output of DMU \( j \), respectively.

Hence, an IDMU and ADMU in DEA are obtained as follows:
\[
\tilde{x}^{\min}_i = (x^{L\min}_i, x^{R\min}_i, \alpha^{L\min}_i, \alpha^{R\min}_i), \quad \forall i
\]
\[
\tilde{x}^{\max}_i = (x^{L\max}_i, x^{R\max}_i, \alpha^{L\max}_i, \alpha^{R\max}_i), \quad \forall i
\]
\[
\tilde{y}^{\min}_r = (y^{L\min}_r, y^{R\min}_r, \beta^{L\min}_r, \beta^{R\min}_r), \quad \forall r
\]
\[
\tilde{y}^{\max}_r = (y^{L\max}_r, y^{R\max}_r, \beta^{L\max}_r, \beta^{R\max}_r), \quad \forall r
\] (8)

where
\[
x^{L\min}_i = \min \{x^{L}_i\}, \quad x^{R\min}_i = \min \{x^{R}_i\},
\alpha^{L\min}_i = \min \{\alpha^{L}_i\}, \quad \alpha^{R\min}_i = \min \{\alpha^{R}_i\},
\]
\[
x^{L\max}_i = \max \{x^{L}_i\}, \quad x^{R\max}_i = \max \{x^{R}_i\},
\alpha^{L\max}_i = \max \{\alpha^{L}_i\}, \quad \alpha^{R\max}_i = \max \{\alpha^{R}_i\},
\]
\[
y^{L\min}_r = \min \{y^{L}_r\}, \quad y^{R\min}_r = \min \{y^{R}_r\},
\beta^{L\min}_r = \min \{\beta^{L}_r\}, \quad \beta^{R\min}_r = \min \{\beta^{R}_r\},
\]
\[
y^{L\max}_r = \max \{y^{L}_r\}, \quad y^{R\max}_r = \max \{y^{R}_r\},
\beta^{L\max}_r = \max \{\beta^{L}_r\}, \quad \beta^{R\max}_r = \max \{\beta^{R}_r\},
\]

Note that a virtual IDMU and virtual ADMU may not exist in practical production activity.

3. Fuzzy DEA models

Suppose that we have \( n \) DMUs. Each DMU \( j \) (\( j = 1, \ldots, n \)) produces \( s \) different outputs, \( y_{rj} \) (\( r = 1, \ldots, s \)), using \( m \) different inputs, \( x_{ij} \) (\( i = 1, \ldots, m \)). DEA models are first developed for crisp data and then extended to fuzzy data. Note
that fuzzy data can be expressed as LR-fuzzy number
\[
\tilde{x}_{ij} = (x^L_{ij}, x^R_{ij}, \alpha^L_{ij}, \beta^R_{ij})_{L_i, R_i}, \quad i=1, \ldots, m, \; j=1, \ldots, n
\]
and
\[
\tilde{y}_{ij} = (y^L_{ij}, y^R_{ij}, \alpha^L_{ij}, \beta^R_{ij})_{L_{ij}, R_{ij}}, \quad r=1, \ldots, s
\]
as the trapezoidal fuzzy numbers where first, second, third and forth components display the left, right, left spread and right spread of the related numbers, respectively. Meanwhile for simplifying, data can be described by means of LR-fuzzy numbers of the same type as follows:
\[
L_{i1} = \ldots = L_{im} = L_i, \quad i = 1, \ldots, m
\]
\[
L'_{r1} = \ldots = L'_{rm} = L'_r, \quad r = 1, \ldots, s
\]
\[
R_{i1} = \ldots = R_{im} = R_i, \quad i = 1, \ldots, m
\]
\[
R'_{r1} = \ldots = R'_{rm} = R'_r, \quad r = 1, \ldots, s
\]
(9)

Let the input-oriented CCR model is used with fuzzy data. Then, we are developed FDEA models with a virtual IDMU and ADMU to get the best and the worst possible relative efficiencies, respectively.

However, the IDMU is a virtual unit, its production behavior should be come the goal of each DMU. According to Wang and Luo (2006), the efficiency of the IDMU can be defined as:

\[
\begin{align*}
\max & \quad \theta_{ID} = \frac{\sum_{i=1}^{m} u_i y_{ij}^{\max}}{\sum_{i=1}^{m} v_i x_{ij}^{\min}} \\
\text{s.t.} & \\
\sum_{i=1}^{m} u_i v_{ij} & \leq 1 \quad \forall j \\
u_r, v_i & \geq 0 \quad \forall r, i
\end{align*}
\]
(10)

where \(u_r\) (\(r = 1, \ldots, s\)) and \(v_i\) (\(i = 1, \ldots, m\)) being the weight on \(r\)th output and \(i\)th input, respectively. Model (10) can be solved through performing the Charnes–Cooper transformation (1962):

\[
\begin{align*}
\max & \quad \theta_{ID} = \sum_{r=1}^{s} u_r y_{rj}^{\max} \\
\text{s.t.} & \\
\sum_{i=1}^{m} u_i x_{ij} & \leq 1 \quad \forall j \\
u_r, v_i & \geq 0 \quad \forall r, i
\end{align*}
\]
(11)

The dual program of (11) is obtained as:

\[
\begin{align*}
\min & \quad \theta_{ID} \\
\text{s.t.} & \\
\sum_{j=1}^{n} \lambda_j x_{ij} & \leq \theta_{ID} x_{ij}^{\min} \quad \forall i \\
\sum_{j=1}^{n} \lambda_j y_{rj} & \geq y_{rj}^{\max} \quad \forall r \\
\lambda_j & \geq 0 \quad \forall j
\end{align*}
\]
(12)

When input-output data are LR-fuzzy number, (12) can be expressed as the following fuzzy LP problem:

\[
\begin{align*}
\min & \quad \theta_{ID} \\
\text{s.t.} & \\
\sum_{j=1}^{n} \lambda_j x_{ij} & \leq \theta_{ID} x_{ij}^{\min} \quad \forall i \\
\sum_{j=1}^{n} \lambda_j y_{rj} & \geq y_{rj}^{\max} \quad \forall r \\
\lambda_j & \geq 0 \quad \forall j
\end{align*}
\]
(13)

where,
\[
\begin{align*}
\bar{x}_{ij} = (x^L_{ij}, x^R_{ij}, \alpha^L_{ij}, \alpha^R_{ij}) \\
\bar{y}_{rj} = (y^L_{rj}, y^R_{rj}, \beta^L_{rj}, \beta^R_{rj}) \\
\bar{x}_{ij} = (x^L_{ij}, x^R_{ij}, \alpha^L_{ij}, \alpha^R_{ij})
\end{align*}
\]
Using inequality (6) and linear combinations as (2), (13) can be transformed as the following equivalent LP model:

\[
\begin{align*}
\min \quad & \theta_{ID} \\
\text{s.t.} \\
\sum_{j=1}^{n} \lambda_{j} x_{ij}^{L} & \leq \theta_{ID} x_{i}^{L} \theta_{min} \quad \forall i \\
\sum_{j=1}^{n} \lambda_{j} x_{ij}^{R} & \leq \theta_{ID} x_{i}^{R} \theta_{min} \quad \forall i \\
\sum_{j=1}^{n} \lambda_{j} x_{ij}^{L} - \sum_{j=1}^{n} \lambda_{j} \alpha_{ij}^{L} & \leq \theta_{ID} x_{i}^{L} \theta_{min} \\
& - \theta_{ID} \alpha_{i}^{L} \theta_{max} \quad \forall i \\
\sum_{j=1}^{n} \lambda_{j} x_{ij}^{R} + \sum_{j=1}^{n} \lambda_{j} \alpha_{ij}^{R} & \leq \theta_{ID} x_{i}^{R} \theta_{min} \\
& + \theta_{ID} \alpha_{i}^{R} \theta_{min} \quad \forall i \\
\sum_{j=1}^{n} \lambda_{j} y_{ij}^{L} & \geq y_{r}^{L} \theta_{max} \quad \forall r \\
\sum_{j=1}^{n} \lambda_{j} y_{ij}^{R} & \geq y_{r}^{R} \theta_{max} \quad \forall r \\
\sum_{j=1}^{n} \lambda_{j} y_{ij}^{L} - \sum_{j=1}^{n} \lambda_{j} \beta_{ij}^{L} & \geq y_{r}^{L} \theta_{max} \\
& - \beta_{r}^{L} \theta_{min} \quad \forall r \\
\sum_{j=1}^{n} \lambda_{j} y_{ij}^{R} + \sum_{j=1}^{n} \lambda_{j} \beta_{ij}^{R} & \geq y_{r}^{R} \theta_{max} \\
& + \beta_{r}^{R} \theta_{max} \quad \forall r \\
\lambda_{j} & \geq 0 \quad \forall j \quad (14)
\end{align*}
\]

Therefore, the optimal value of (14) provides an evaluation of the efficiency of IDMU in which all the possible values of the different variables for all the DMUs at all the possibility levels are considered. Model (15) is utilized to determine the best relative efficiency of DMU \( p \) using the efficiency of IDMU (with exact data) (Wang and Luo, 2006):

\[
\begin{align*}
\max \quad & \theta_{p} = \frac{\sum_{r=1}^{s} u_{r} y_{rp}}{\sum_{i=1}^{m} v_{i} x_{ip}} \\
\text{s.t.} \\
\sum_{r=1}^{s} u_{r} y_{r}^{max} & \geq \theta_{ID}^{*} \\
\sum_{i=1}^{m} v_{i} x_{ij}^{min} & \leq 1 \quad \forall j \\
u_{r}, v_{i} & \geq 0 \quad \forall r, i \quad (15)
\end{align*}
\]

Model (15) can be expressed as the following linear programming problem:

\[
\begin{align*}
\max \quad & \theta_{p} = \sum_{r=1}^{s} u_{r} y_{rp} \\
\text{s.t.} \\
\sum_{i=1}^{m} v_{i} x_{ip} & = 1 \\
\sum_{r=1}^{s} u_{r} y_{r}^{max} - \sum_{i=1}^{m} v_{i} x_{ij}^{min} & \geq 0 \\
\sum_{r=1}^{s} u_{r} y_{r}^{max} - \sum_{i=1}^{m} v_{i} x_{ij}^{min} & \leq 0 \quad \forall j \\
u_{r}, v_{i} & \geq 0 \quad \forall r, i \quad (16)
\end{align*}
\]

The dual program of (16) is as follows:

\[
\begin{align*}
\min \quad & \theta_{p} \\
\text{s.t.} \\
\lambda_{j} & \geq 0 \quad \forall j \quad (14)
\end{align*}
\]

\[
\tilde{y}_{ij} = (y_{ij}^{L}, y_{ij}^{R}, \beta_{ij}^{L}, \beta_{ij}^{R})
\]
\[ \theta_{ID}^* x_{i}^{\min} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_p x_{ip} \quad \forall i \]

\[ y_r^{\max} \lambda^* + \sum_{j=1}^{n} \lambda_j y_{nj} \geq y_{np} \quad \forall r \]

\[ \lambda^* \leq 0 \]

\[ \lambda_j \geq 0 \quad \forall j \quad (17) \]

Now, since input-output data are LR-fuzzy number and \( \theta_{ID}^* \) is taken from (14), the best relative efficiency of DMU \( p \) can be determined through the following model:

\[
\begin{align*}
\text{Min} & \quad \theta_p \\
\text{S.t.} & \quad \theta_{ID}^* x_{i}^{\min} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_p x_{ip} \quad \forall i \\
& \quad y_r^{\max} \lambda^* + \sum_{j=1}^{n} \lambda_j y_{nj} \geq y_{np} \quad \forall r \\
& \quad \lambda^* \leq 0 \\
& \quad \lambda_j \geq 0 \quad \forall j \\
(18)
\end{align*}
\]

Similarly, suppose that we have:

\[
\begin{align*}
\tilde{y}_{ij} &= (y_{ij}^L, y_{ij}^R, \beta_{ij}^L, \beta_{ij}^R) \\
\tilde{y}_{r}^{\max} &= (y_{r}^{L \max}, y_{r}^{R \max}, \beta_{r}^{L \min}, \beta_{r}^{R \max}) \\
\tilde{x}_{i}^{\min} &= (x_{i}^{L \min}, x_{i}^{R \min}, \alpha_{i}^{L \max}, \alpha_{i}^{R \min}) \\
\tilde{x}_{ij} &= (x_{ij}^L, x_{ij}^R, \alpha_{ij}^L, \alpha_{ij}^R)
\end{align*}
\]

Therefore, (18) can be transformed as the following equivalent LP model:

\[
\begin{align*}
\text{Min} & \quad \theta_p \\
\text{S.t.} & \quad \theta_{ID}^* x_{i}^{\min} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta_p x_{ip} \quad \forall i \\
& \quad y_r^{\max} \lambda^* + \sum_{j=1}^{n} \lambda_j y_{nj} \geq y_{np} \quad \forall r \\
& \quad \lambda^* \leq 0 \\
& \quad \lambda_j \geq 0 \quad \forall j \\
(19)
\end{align*}
\]
Max $\phi_{AD}$

S.t.

$$\sum_{j=1}^{n} \lambda_j x_{ij} \geq \phi_{AD} x_{i}^{\max} \quad \forall i$$

$$\sum_{j=1}^{n} \lambda_j y_{ij} \leq y_{r}^{\max} \quad \forall r$$

$\lambda_j \geq 0 \quad \forall j$ \hspace{1cm} (20)

Similarly, (20) can be obtained by the following LP:

Max $\phi_{AD}$

S.t.

$$\sum_{j=1}^{n} \lambda_j x_{ij} \geq \phi_{AD} x_{i}^{L \max} \quad \forall i$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} \geq \phi_{AD} x_{i}^{R \max} \quad \forall i$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{L} - \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{L} \geq \phi_{AD} x_{i}^{L \max}$$

$$-\phi_{AD} \alpha_{i}^{L \min} \quad \forall i$$

$$\sum_{j=1}^{n} \lambda_j x_{ij}^{R} + \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{R} \geq \phi_{AD} x_{i}^{R \max}$$

$$+\phi_{AD} \alpha_{i}^{R \max} \quad \forall i$$

$$\sum_{j=1}^{n} \lambda_j y_{ij}^{L} \leq y_{r}^{L \min} \quad \forall r$$

$$\sum_{j=1}^{n} \lambda_j y_{ij}^{R} \leq y_{r}^{R \min} \quad \forall r$$

$$\sum_{j=1}^{n} \lambda_j y_{ij}^{L} - \sum_{j=1}^{n} \lambda_j \beta_{ij}^{L} \leq y_{r}^{L \min} + \beta_{r}^{L \max} \quad \forall r$$

Let $\phi_{AD}^{*}$ be the worst efficiency of the ADMU with fuzzy data, its efficiency is clearly worse than other units. Model (22) is utilized to determine the worse relative efficiency of DMU $p$ using the efficiency of ADMU (with exact data) (Wang and Luo, 2006):

Max $\phi_{p} = \frac{\sum_{r=1}^{s} u_{r} y_{r p}}{\sum_{i=1}^{m} v_{i} x_{i p}}$

S.t.

$$\sum_{r=1}^{s} u_{r} y_{r}^{\min} \leq \phi_{AD}^{*}$$

$$\sum_{i=1}^{m} v_{i} x_{i}^{\max} \geq 1 \quad \forall j$$

$$u_{r}, v_{i}, \geq 0 \quad \forall r, i$$ \hspace{1cm} (22)

The fractional programming problem (22) can be expressed as the following linear programming:

Max $\phi_{p} = \sum_{r=1}^{s} u_{r} y_{r p}$

S.t.
\[
\sum_{i=1}^{m} v_i x_{ip} = 1
\]
\[
\sum_{r=1}^{s} u_r y_{min} r - \sum_{i=1}^{m} v_i \varphi^*_{AD} x_{i}^{\max} \leq 0
\]
\[
\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \geq 0 \quad \forall j
\]
\[
u_r, v_i \geq 0 \quad \forall r, i
\] (23)

The dual of the above model can be formulated as follows:

**Max** $\varphi_p$

**S.t.**

\[
\varphi^*_{AD} x_i^{\max} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij} \geq \varphi_p x_{ip} \quad \forall i
\]
\[
y_r^{\min} \lambda^* + \sum_{j=1}^{n} \lambda_j y_{rj} \leq y_{np} \quad \forall r
\]
\[
\lambda^* \leq 0
\]
\[
\lambda_j \geq 0 \quad \forall j
\] (24)

Now, since input-output data are LR-fuzzy number and $\varphi^*_{ID}$ is taken from (21), the worst possible relative efficiency of DMU $p$ can be determined through the following model:

**Max** $\varphi_p$

**S.t.**

\[
\varphi^*_{ID} x_i^{L \max} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij} \geq \varphi_p x_{ip} \quad \forall i
\]
\[
\varphi^*_{ID} x_i^{R \max} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij} \geq \varphi_p x_{ip} \quad \forall i
\]
\[
\varphi^*_{AD} x_i^{L \max} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij}^{L} - \varphi^*_{AD} x_i^{L \min} \lambda^*
\]
\[
- \sum_{j=1}^{n} \lambda_j \alpha_{ij} \geq \varphi_p x_{ip}^{L} - \varphi_p x_{ip}^{L} \quad \forall i
\]
\[
\varphi^*_{AD} x_i^{R \max} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij}^{R} + \varphi^*_{AD} x_i^{R \max} \lambda^*
\]
\[
+ \sum_{j=1}^{n} \lambda_j \alpha_{ij} \geq \varphi_p x_{ip}^{R} + \varphi_p x_{ip}^{R} \quad \forall i
\]
\[
y_r^{L \min} \lambda^* + \sum_{j=1}^{n} \lambda_j y_{rj} \leq y_{np}^{L} \quad \forall r
\]
\[
y_r^{R \min} \lambda^* + \sum_{j=1}^{n} \lambda_j y_{rj} \leq y_{np}^{R} \quad \forall r
\]
\[
y_r^{L \min} \lambda^* + \sum_{j=1}^{n} \lambda_j y_{rj} - \beta_{r}^{L \max} \lambda^*
\]
\[
- \sum_{j=1}^{n} \lambda_j \beta_{r} \leq y_{np}^{L} - \beta_{r}^{L \min} \lambda^* \quad \forall r
\] (25)

where

\[
y_{rj} = (y_{rj}^{L}, y_{rj}^{R}, \beta_{rj}^{L}, \beta_{rj}^{R})
\]
\[
\tilde{x}_{ij} = (x_{ij}^{L}, x_{ij}^{R}, \alpha_{ij}^{L}, \alpha_{ij}^{R})
\]
\[
\tilde{x}_{ij}^{\max} = (x_{ij}^{L \max}, x_{ij}^{R \max}, \alpha_{ij}^{L \min}, \alpha_{ij}^{R \max})
\]
\[
\tilde{y}_{rj} = (y_{rj}^{L \min}, y_{rj}^{R}, \beta_{rj}^{L \max}, \beta_{rj}^{R \min})
\]

Similarly, model (25) can be transformed as the following equivalent LP model:

**Max** $\varphi_p$

**S.t.**

\[
\varphi^*_{ID} x_i^{L \max} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij}^{L} \geq \varphi_p x_{ip}^{L} \quad \forall i
\]
\[
\varphi^*_{ID} x_i^{R \max} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij}^{R} \geq \varphi_p x_{ip}^{R} \quad \forall i
\]
\[
\varphi^*_{AD} x_i^{L \max} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij}^{L} - \varphi^*_{AD} x_i^{L \min} \lambda^*
\]
\[
- \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{L} \geq \varphi_p x_{ip}^{L} - \varphi_p x_{ip}^{L} \quad \forall i
\]
\[
\varphi^*_{AD} x_i^{R \max} \lambda^* + \sum_{j=1}^{n} \lambda_j x_{ij}^{R} + \varphi^*_{AD} x_i^{R \max} \lambda^*
\]
\[
+ \sum_{j=1}^{n} \lambda_j \alpha_{ij}^{R} \geq \varphi_p x_{ip}^{R} + \varphi_p x_{ip}^{R} \quad \forall i
\]
\[
y_r^{L \min} \lambda^* + \sum_{j=1}^{n} \lambda_j y_{rj} \leq y_{np}^{L} \quad \forall r
\]
\[
y_r^{R \min} \lambda^* + \sum_{j=1}^{n} \lambda_j y_{rj} \leq y_{np}^{R} \quad \forall r
\]
\[
y_r^{L \min} \lambda^* + \sum_{j=1}^{n} \lambda_j y_{rj} - \beta_{r}^{L \max} \lambda^*
\]
\[
- \sum_{j=1}^{n} \lambda_j \beta_{r} \leq y_{np}^{L} - \beta_{r}^{L \min} \lambda^* \quad \forall r
\] (25)
Ideal and anti-ideal decision making units: A fuzzy...

\[
y^R_y - \min_{r} \lambda^* + \sum_{j=1}^{n} \lambda_j y^R_{ij} + \beta^R_{R} \min_{r} \lambda^*
\]
\[
+ \sum_{j=1}^{n} \lambda_j \beta^R_{ij} \leq y^R_{mp} + \beta^R_{mp} \quad \forall r
\]
\[
\lambda^* \leq 0
\]
\[
\lambda_j \geq 0 \quad \forall j
\]  

(26)

Let \( \theta_j^* \) and \( \varphi_j^* \) be the best and the worst possible relative efficiencies of DMU \( j \), respectively. These two distinctive efficiency assessments may lead to completely different results. Hence, it is essential to consider them together to give an overall assessment of each DMU. Wang and Luo (2006) propose that the coefficient closeness value should be calculated in the similar manner as used in the technique for order preference by similarity to ideal solution (TOPSIS). Coefficient closeness considers the distances from the ideal (to be minimized) and for the nadir (to be maximized) simultaneously through the TOPSIS formula as follows:

\[
CC_j = \frac{\varphi^*_j - \varphi^*_AD}{(\varphi^*_j - \varphi^*_AD) + (\theta^*_D - \theta^*_j)}
\]  

(27)

Obviously, \( 0 \leq CC_j \leq 1 \) where \( j = 1, 2, ..., n \). The bigger difference between \( \varphi^*_j \) and \( \varphi^*_AD \) and the smaller difference between \( \theta^*_D \) and \( \theta^*_j \) mean the better the performance of DMU \( j \). Note that overall ranking for all DMUs provides in uncertain environment based on the \( CC_j \) index.

In the next section, a numerical experiment is presented to illustrate the proposed approach. It is obvious, if inputs and outputs are assumed to be symmetrical triangular membership functions, then the two constraints associated with the main values reduce to only one, and can be eliminated since it is redundant as a consequence of the symmetry.

4. Numerical example

A simple numerical example with fuzzy single-input and single-output was introduced by Leon et al. (2003). We will consider this example with its data listed in Table 1 and it should be noted, each DMU consumes a symmetrical triangular fuzzy input to produce a symmetrical triangular fuzzy output. The efficiencies of DMUs with proposed method in Leon et al. (2003) are listed in last column of Table 1.

From Table 1 we see A, B, C and G are efficient with proposed method in Leon et al. (2003). Their final result for efficient units is the fuzzy set as follows:

\[
\tilde{E}_f = \{(A, 1), (B, 0.3), (C, 1), (G, 0.9)\}
\]

Now, we use our proposed models with virtual IDMU and ADMU for these DMUs. The IDMU and ADMU are shown in the last two rows of Table 1. The resulting efficiency scores and \( CC \) values are listed in Table 2.

From Table 2 that the DEA models based on IDMU evaluations DMUs A and C are efficient, while other DMUs are not efficient. On the other hand, is based on ADMU appraisals C is the best DMU, which is followed by DMUs A, B, D, E and G, respectively, and both F and H are the worst DMU. When IDMU and ADMU efficiencies obtained are aggregated with \( CC \) index for each DMU, a fully ranking order is achieved that are shown in the fourth and fifth columns of Table 2. The overall ranking from proposed models is nearly like to result of Leon et al. such as A and C that are important for analyst because they can be pattern for future variation in input-output data of other DMUs.

5. Conclusions

In this paper, two kinds of fuzzy DEA models based on virtual IDMU and ADMU are suggested for evaluating the efficiencies of DMUs with fuzzy input and output data. The two distinctive efficiencies are integrated using a closeness coefficient index. Using the obtained \( CC \) index for each DMU, the overall ranking can be provided. The proposed models for performance assessment of DMUs in fuzzy environment can be applied easily in analyzing management evaluation problem of units. It can be concluded that the proposed fuzzy DEA models extend BCC model to more general forms where crisp, fuzzy and hybrid data can be handled easily. Because uncertainty always exists in human thinking and judgment, fuzzy DEA models can play an important role for perceptual evaluation problems comprehensively existing in the real world.
Table 1. Fuzzy data of DMUs in numerical example and Leon et al. (2003) results.

<table>
<thead>
<tr>
<th>DMUs</th>
<th>Input</th>
<th>Output</th>
<th>Loen et al. efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(3, 2, 2)</td>
<td>(3, 1, 1)</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>(4, 0.5, 0.5)</td>
<td>(2.5, 1, 1)</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>(4.5, 1.5, 1.5)</td>
<td>(6, 1, 1)</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>(6.5, 0.5, 0.5)</td>
<td>(4.125, 1.25)</td>
<td>0.75</td>
</tr>
<tr>
<td>E</td>
<td>(7, 2, 2)</td>
<td>(5, 0.5, 0.5)</td>
<td>0.6429</td>
</tr>
<tr>
<td>F</td>
<td>(8, 0.5, 0.5)</td>
<td>(3.5, 0.5, 0.5)</td>
<td>0.605</td>
</tr>
<tr>
<td>G</td>
<td>(10, 1, 1)</td>
<td>(6, 0.5, 0.5)</td>
<td>1</td>
</tr>
<tr>
<td>H</td>
<td>(6, 0.5, 0.5)</td>
<td>(2, 1.5, 1.5)</td>
<td>0.6923</td>
</tr>
<tr>
<td>IDMU</td>
<td>(3, 2, 0.5)</td>
<td>(6, 0.5, 1.5)</td>
<td>-</td>
</tr>
<tr>
<td>ADMU</td>
<td>(10, 0.5, 2)</td>
<td>(2, 1.5, 0.5)</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2. The efficiencies by proposed models.

<table>
<thead>
<tr>
<th>DMUs</th>
<th>IDMU efficiency</th>
<th>ADMU efficiency</th>
<th>CC</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1.6118</td>
<td>0.379</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>0.6667</td>
<td>1.5425</td>
<td>0.330</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>2.4608</td>
<td>0.508</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>0.6429</td>
<td>1.5189</td>
<td>0.323</td>
<td>4</td>
</tr>
<tr>
<td>E</td>
<td>0.6</td>
<td>1.2986</td>
<td>0.274</td>
<td>5</td>
</tr>
<tr>
<td>F</td>
<td>0.4235</td>
<td>1</td>
<td>0.191</td>
<td>7</td>
</tr>
<tr>
<td>G</td>
<td>0.6</td>
<td>1.2557</td>
<td>0.265</td>
<td>6</td>
</tr>
<tr>
<td>H</td>
<td>0.4615</td>
<td>1</td>
<td>0.193</td>
<td>8</td>
</tr>
<tr>
<td>IDMU</td>
<td>3</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>ADMU</td>
<td>-</td>
<td>0.3922</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>
References


