

# A new Simulated Annealing algorithm for the robust coloring problem

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## Abstract

The Robust Coloring Problem (RCP) is a generalization of the well-known Graph Coloring Problem where we seek for a solution that remains valid when extra edges are added. The RCP is used in scheduling of events with possible last-minute changes and study frequency assignments of the electromagnetic spectrum. This problem has been proved as NP-hard and in instances larger than 30 vertices, meta-heuristics are required. In this paper a Simulated Annealing Algorithm is proposed, and his performance is compared against other techniques such as GRASP, Tabu Search and Scatter Search. In the classic instances of the problem our proposal method which gives the best solutions at this moment.

**Keywords:** Robust coloring problem; Graph coloring; Heuristics; Simulated Annealing

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## 1. Introduction

In the RCP two graphs are considered, the original, made by the edges that indicate the incompatibilities at the present time that have two users (vertices) of some resource (the color assigned to the vertex); in this graph we must find a valid coloring. In the other hand, we got a complementary graph, made by all those edges not included in the original graph; these complementary edges have a penalization (for instance, the probability that some new edge can be included in the original graph on a close future). If two vertices have the same color and are adjacent to some complementary edge, the total penalization is the sum of those penalizations. This is called rigidity associated to this particular coloring.

This paper is presented as follows: in Section 2 we present an illustrative example of the problem. Section 3 describes mathematically the RCP. Section 4

details the Simulated Annealing (SA) Technique. Section 5 describes the SA algorithm proposed; Section 6 shows a comparative study of previous algorithms against SA. Finally we present the conclusions and some final recommendations.

## 2. Assignment of frequencies in the electromagnetic spectrum: an application of the RCP

In the times we live there is a big demand in the use of electromagnetic frequencies (wireless internet, cell phones, GPS transmitters in transportation trucks, etc). For this reason, the number of users which require the service of these assigned frequencies is increasing day by day. The robust coloring problem can be used to minimize the changes of frequencies required for a user when it goes from one receptor base to another, in order to reduce supervision and inter-

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ruptions in the service. Suppose that we have 5 cell phones sharing certain geographic region, like it is shown in the Figure 1.

In the left side Figure 1, vertices represent active cell phones. At this moment, every edge represents when two cell phones are close enough to cause interference between them, and each color represents a different frequency. The complementary edges (right side of Figure 1) represents, for example, the probability of two cell phones to interfere in a certain period of time

In left Figure 1, we can appreciate that the graph can be validly colored with  $k=3$  colors. At the right of the same figure, the vertices painted with color "2" add 0.3 to the rigidity (because that is the penalization associated at the edge that join them), also, the vertices painted with color "1" are adjacent with a penalization of 0.5; finally, the total penalization, also called rigidity, is the sum of the associated penalizations of edges with adjacent vertices of the same color  $i$  ( $i = 1,2,3$ ), in this case  $R(c) = 0.3 + 0.5 = 0.8$ , which is the associated rigidity at this coloring  $c$  of the graphs.

Another valid coloring with a smaller rigidity ( $R(c) = 0.3 + 0.2 = 0.5$ ), is shown in Figure 2.

If we are allowed to use four colors instead of three, can be found a valid coloring with  $R(c) = 0.2$  as we show in Figure 3.

In the next section the concepts illustrated in this example are presented in formal approach.

### 3. The robust coloring problem: Definitions

**Complementary Graph.** Given an original graph  $G(V, E)$ , where  $V$  is the set of vertices and  $E$  is the set of edges, the complementary graph is given by  $\overline{G}(V, \overline{E})$ , where  $\overline{E}$  is the set of complementary edges:

$$\{i, j\} \in \overline{E} \Leftrightarrow \{i, j\} \notin E. \quad (1)$$

**Valid Coloring.** Given an original graph  $G(V, E)$  and  $k$  a positive integer, with number of vertices and edges  $|V| = n$  y  $|E| = m$  respectively,  $k$  is the number of colors that will paint the graphs, which is bigger or equal to the chromatic number of the original graph  $\chi(G)$ . Let  $c^k$  be a  $k$ -coloring, meaning:

$$c^k : V \rightarrow \{1, 2, \dots, k\}, \quad (2)$$

and verifying:

$$c^k(i) \neq c^k(j) \quad \forall \{i, j\} \in E, \quad (3)$$

then  $c^k$  is a valid coloring of the graph  $G$ .

**Rigidity.** Given a complementary graph  $\overline{G}$ , and a known family of penalizations  $\{p_{ij} \geq 0, \{i, j\} \in \overline{E}\}$ , the rigidity level  $R(c^k)$  of the  $k$ -coloring  $c^k$  is defined as the sum of the penalties of complementary edges  $\overline{E}$ , which extremes are equally colored:

$$R(c^k) = \sum_{\{i, j\} \in \overline{E}, c^k(i)=c^k(j)} p_{ij}. \quad (4)$$

**Robust Coloring Problem.** Given a graph  $G(V, E)$ , with a number of colors  $k \geq \chi(G)$ , and a penalty matrix defined on the complementary edge set  $\{p_{ij}, \{i, j\} \in \overline{E}\}$ , the Robust Coloring Problem consists in finding the  $k$ -coloring  $c_R^k$  with the least rigidity level:

$$R(c_R^k) = \min_{c^k} R(c^k). \quad (5)$$

This means that for a fixed  $k$ , the less the rigidity level, the more robust the coloring

### 4. The Simulated Annealing technique

The Simulated Annealing (SA) technique was proposed originally by Kirkpatrick, Gelatt and Vecchi [2], has been used to successfully solve many other NP problems, was inspired in the concept of annealing on materials in industry. This technique uses the controlled heating and cooling of a material to increase the size of crystals and the reduction of its defects. The heating phase causes the atoms to get unbiased from their initial positions (a local minimum of the internal heat in the material) and make them move to other states of internal energy, in search of a new state with less energy.

The slow cooling of the material gives a better chance to find configurations with less internal energy than the initial state. The initial annealing temperature depends of the material, the present state of deformation and his future use.

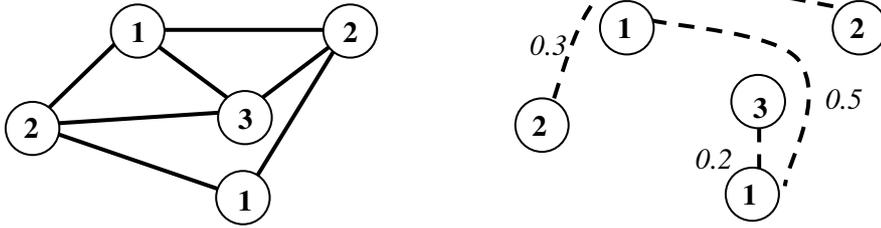


Figure 1. An example of configuration for the RCP, for active cell phones with 3 colors.

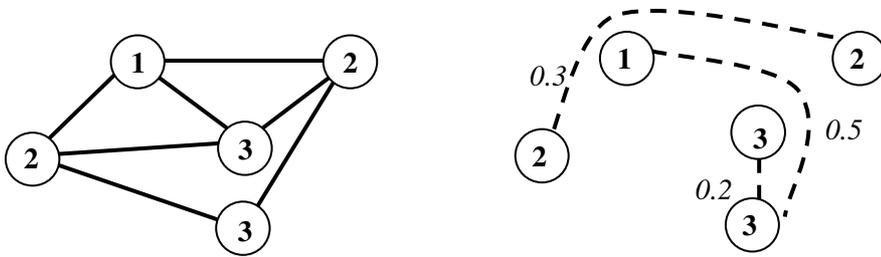


Figure 2. Another example of configuration for the RCP, for active cell phones with 3 colors.

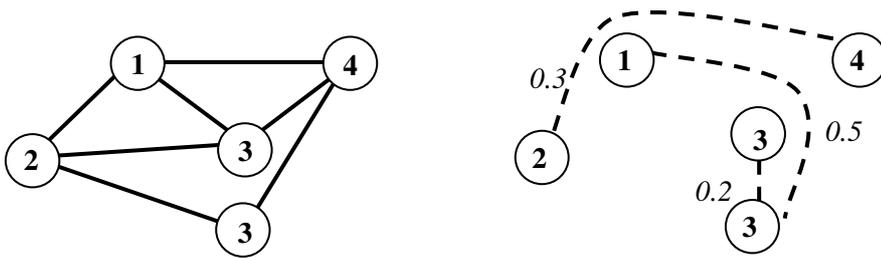


Figure 3. Another example of configuration for the RCP, for active cell phones, with 4 colors.

Followed by the phase of heating there is a process of cooling where the temperature is lowered down slowly until thermodynamic equilibrium is reached. In this way, when the temperature is lowered, the particles are rearranged in states of lower energy until we get a solid with his particles arranged in a crystal-line array, that is, with all his atoms regularly ordered. In a state of perfect crystal, the solid is in his state of lowest internal energy.

We can use an analogy of the industrial annealing process to describe a succession of solutions of a combinatorial optimization problem. Each solution of this problem is analog to a state of the physical system; and his associated objective function (cost-benefit) is equivalent to the energy of a system state. A control parameter is introduced into the combinatorial problem which is analog to the temperature of the solid.

In the implementation of an algorithm of SA a function  $H$  is minimized. The algorithm iteratively proposes changes and such changes are whether or not accepted using the Metropolis Criteria [5]: given two solutions one initial  $i$  and one proposed,  $j$ , if the cost function diminish, (i. e.  $\Delta H = H(i) - H(j) > 0$ ) the new solution is simply accepted.

In the other case the solution is accepted with a probability of  $\exp(\Delta H/T)$ . In every iteration, temperature  $T$  diminish in a  $\beta$  factor (for example, between 0.95 and 0.99) and the number of solutions evaluated in every iteration is increased in an  $\alpha$  factor.

### 5. An algorithm of Simulated Annealing for the robust coloring problem

For this algorithm we considered a cost function associated to each particular coloring  $c$ , made by the number of incompatible edges in  $G(I(c))$  and the rigidity associated to  $\bar{G}$ , given by  $R(c)$ . This way, the cost of every coloring is given by:

$$H(c) = I(c) + \kappa R(c). \quad (6)$$

In this function a pondering parameter,  $\kappa$ , is considered. This parameter weights the relative importance between incompatible edges in  $G$  and the rigidity associated to  $\bar{G}$  for that specific coloring.

The algorithm of Simulated Annealing for the RCP is as follows:

#### Begin

Generate an initial random coloring  $i$ .

Evaluate the cost function:  $H(i) = I(i) + \kappa R(i)$

Set  $T = (n)^{1/2}$ ;  $S = n$ ;  $\beta = 0.95$ ;  $\alpha = 1.05$

**While**  $T > \text{"Minimal Temperature"}$

**For**  $l = 1$  to  $S$

Generate a solution neighboring  $j$ .

If  $\Delta H = H(i) - H(j) > 0$ , then  $j$  becomes the best new solution  $i$ .

Otherwise, generate a random number  $U(0,1)$ . If  $\exp(\Delta H/T)$  is bigger than such random number,

then  $j$  becomes the best new solution  $i$ .

Adjust the parameter:  $S = \alpha * S$

**End** {For}

Adjust the parameter:  $T = \beta * T$

**End** {While}

**End** {Begin}

## 6. Computational results and comparative table

There is a set of well known random instances between 20 y 120 vertices used in previous papers. The best solutions found using these instances, valid coloring with least rigidity for each algorithm are shown in Table 1 and his computer time.

The size of forbidden movements list in the Tabu Search Algorithm was 10, with 500 iterations. The GRASP was executed 100 times and the best solution is presented. In the Scatter Search Algorithm [3], 100 initial solutions were taken, 50/50 between quality and diversity, 5 elements in the *RefSet* and 5 steps in path relinking. In the Simulated Annealing Algorithm are considered  $\alpha=0.99$  and in each generation  $n$  are tested  $\exp(2/\alpha^n)$  solutions.

In 18 of the 22 instances the SA Algorithm found better or at least the same best solution known and only in four cases the best solution still belongs to the Scatter Search Algorithm. Considering the times of execution, SA is relatively slow in small instances but can more robust solutions and is faster than the others for larger instances.

**Table 1.** Comparison between previous algorithms against Simulated Annealing (Rigidity/Time) for the RCP. In the table,  $G_{n,0.5}$  refers to certain graph with  $n$  vertices, and 0.5 of penalization;  $k$  is the number of colors used in the graph;  $al(n)$ ,  $nf/>(1000)$  means that couldn't be found a solution in less than 1000 seconds.

Instance			Rigidity in valid coloring / Computer time (secs.)			
$G_{n,0.5}$	$n$	$k$	Tabu Search	GRASP	Scatter Search	Simulated Annealing
$al(20)$	20	7	7.0970 / 2	7.1423 / 0.14	6.9046 / 1.3	<u>6.6796</u> / 8.26
		8	4.7710 / 3	4.6934 / 0.15	4.6934 / 1.3	<u>3.5934</u> / 5.67
$al(30)$	30	10	8.0623 / 5	7.5749 / 0.44	<u>7.5749</u> / 3.6	<u>7.5749</u> / 8.17
		11	6.0565 / 9	5.9318 / 0.49	<u>5.8890</u> / 3.6	<u>5.8890</u> / 6.57
$al(40)$	40	14	7.1709 / 15	7.3950 / 1.0	7.0837 / 8.5	<u>6.9901</u> / 21.49
		15	5.8173 / 14	6.3117 / 1.0	5.6708 / 7.2	<u>4.9947</u> / 19.29
$al(50)$	50	17	9.8259 / 33	8.9531 / 1.9	<u>8.2587</u> / 13	<u>8.2587</u> / 15.89
		18	7.4966 / 33	7.1464 / 1.9	<u>6.7164</u> / 12	<u>6.7164</u> / 24.47
$al(60)$	60	20	9.8331 / 69	9.9687 / 3.3	8.8676 / 19	<u>8.7568</u> / 23.03
		21	8.2181 / 69	8.1430 / 3.4	7.2380 / 20	<u>7.2048</u> / 23.30
$al(70)$	70	24	11.1307 / 128	11.2388 / 5.3	<u>9.2634</u> / 36	9.3555 / 35.02
		25	9.5478 / 128	9.2145 / 5.5	7.7048 / 33	<u>7.3799</u> / 50.33
$al(80)$	80	27	11.1946 / 218	11.7512 / 8.0	9.9835 / 52	<u>9.7132</u> / 75.41
		28	10.5845 / 219	10.2631 / 8.2	<u>8.5961</u> / 43	8.6118 / 52.42
$al(90)$	90	30	12.2832 / 350	13.4919 / 11.5	10.8911 / 47	<u>10.6400</u> / 41.2
		31	11.3699 / 349	11.5060 / 11.9	<u>9.5008</u> / 89	9.5570 / 60.9
$al(100)$	100	34	12.1932 / 544	12.8675 / 15.8	10.0470 / 123	<u>9.9658</u> / 99.59
		35	12.0650 / 885	11.1317 / 15.6	9.4259 / 124	<u>9.0303</u> / 78.0
$al(110)$	110	37	$nf/>(1000)$	12.7681 / 20.7	10.8463 / 149	<u>10.6524</u> / 152.2
		38	$nf/>(1000)$	11.6574 / 20.2	9.9558 / 171	<u>9.8614</u> / 115.7
$al(120)$	120	40	$nf/>(1000)$	15.0014 / 26.8	11.3507 / 190	<u>11.1545</u> / 144
		41	$nf/>(1000)$	13.5266 / 27.5	<u>10.1258</u> / 191	10.1307 / 150

## 7. Conclusion

In this paper the Robust Coloring Problem was described including an application of it in the assignment of frequencies in the Electromagnetic Spectrum. This problem is of the *NP*-hard kind, and for instances bigger than 30 vertices exact solutions couldn't be found in "reasonable" times. A new Simulated Annealing Algorithm was proposed and was tested in known instances from 20 to 120 vertices. The solutions obtained with this algorithm were compared against other heuristics and in almost every case a similar or better solution was found, in less computer time.

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