

# Numerical modeling of economic uncertainty

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## Abstract

Representation and modeling of economic uncertainty is addressed by different modeling methods, namely stochastic variables and probabilities, interval analysis, and fuzzy numbers, in particular triple estimates. Focusing on discounted cash flow analysis numerical results are presented, comparisons are made between alternative modeling methods, and characteristics of the methods are discussed.

**Keywords:** Economic uncertainty; Modeling; Stochastic; Probability; Interval; Fuzzy number; Triple estimate; Discounted cash flow

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## 1. Introduction

This paper deals with an intricate issue in economic theory, however, from a practical point of view. The larger problem area under consideration is that of choice under risk and uncertainty and the practical perspective is that of modeling of economic uncertainty in practical decision situations.

Central to the modeling of economic uncertainty is the way in which uncertainty is actually represented numerically as a meaningful reflection of the characteristics of uncertainty present. Further, the way of actually processing the uncertain input variables so that additional uncertainty is not introduced and finally the interpretation and communication of the model output variables as a basis for rational decision making is important.

The notion of risk and uncertainty being relevant for economic analysis was suggested by Knight [11] and the concepts were incorporated into economic theory by von Neumann and Morgenstern [21] who developed a rational foundation and rules for decision making according to expected utility, see also Hertz [7]. As far as risk and uncertainty in technology management is concerned Kyläheiko [12] made an extensive study focusing on economic theory and methodology.

The distinction made by Knight [11] between risk (the agent can assign mathematical probabilities to the randomness of the decision situation) and uncertainty (the actor cannot assign probabilities) has later been disputed by economists arguing that they are

really representing one and the same thing. This debate is long running and far from being resolved at present. In this paper we shall refer to uncertainty not in the Knightian way but rather in a more general sense that allows us to refer to uncertain economic variables by means of a variety of different representations.

The traditional approach to representation of uncertainty in economic theory is that of probabilities. An uncertain variable may be represented by a probability distribution reflecting either the objective nature of the variable or the subjective belief of the agent. The most common objectivist position argues that the probability of a particular event in a particular trial is the relative frequency of occurrence of that event in an infinite sequence of similar trials. Obviously, the idea of infinite repetition is referring to an idealized laboratory experiment like rolling an ideal dice an infinite number of times. How then, is one to comprehend the probability of one-of-a-kind-events, such as the probability of a quote leading to an order?

Consequently, there have been many objections to this view of probability arguing that randomness is not an objectively measurable phenomenon but rather a knowledge phenomenon. Thus probability is rather an epistemological and not an ontological issue. This epistemic or knowledge view of probability can be traced back to Bayes [1] and Laplace [13]. More recently Ramsey [16] asserted that probability is related to the knowledge possessed by a particular individual and thus probability represents personal belief rather than objective knowledge.

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We shall not take this discussion further but merely state that the position taken in this paper is to regard probabilities to be a way of representing uncertain knowledge. Accordingly, the numerical value of a probability is interpreted as being proportional to the sum of money a rational individual would be willing to pay should a proposition one asserts prove false. The measure of uncertainty so defined can be shown to obey the axioms of probability theory.

Probability theory and statistics today represent a well established mathematical theory with clear axioms and has reached an advanced stage of development. Since criticism has been raised towards probability theory as being a too normative framework to take all the aspects of uncertain judgment into account, Dubois and Prade [5], we will investigate alternative methods of modeling economic uncertainties like the interval representation and the fuzzy number representation.

The interval representation, Moore [14], lends itself to a situation where the knowledge of an uncertain parameter is limited to knowing its minimum and maximum value whereas nothing else is known. Based upon a mathematical theory of interval analysis this approach has shown to be useful in keeping track of worst and best cases in economic analyses and thus contribute to improved decision processes.

The concept of fuzzy set was introduced by Zadeh [22] for the purpose of modeling the imprecision and ambiguity of ordinary language. It is based on the concept of possibility rather than probability and translates natural language expressions into the mathematical formalism of possibility measures. It is widely recognized that possibility is distinct from probability. As mentioned earlier probabilities can be interpreted as relative frequencies or, more generally, imprecise knowledge or belief whereas possibility relates to the degree of feasibility and ease of attainment. In this paper the meaning of possibility will be intuitively clear from the particular applications presented.

**2. Modeling by stochastic numbers and probabilities**

A stochastic variable  $X=\{\mu; \sigma\}$  is characterized by its expected value  $E(X)=\mu$  and variance  $VAR(X)=\sigma^2$ , where  $\sigma$  is the standard deviation. As an example  $\{1.000;100\}$  denotes the uncertain volume of sales during the next budgetary period, meaning that the sales volume is expected to be 1.000 with a standard deviation of 100.

Let  $X_1$  and  $X_2$  be independent stochastic variables with expected values  $E(X_1)=\mu_1$  and  $E(X_2)=\mu_2$  and variances  $VAR(X_1) = \sigma_1^2$  and  $VAR(X_2) = \sigma_2^2$ . It may be shown that basic calculations may be carried out according to the formulas shown in Table 1.

In the general case of  $Y$  being a function of  $m$  independent stochastic variables as:

$$Y = Y\{X_1, X_2, \dots, X_m\}, \tag{1}$$

we can approximate  $Y$  by means of a Taylor series (ignoring second and higher order terms):

$$Y \cong Y(\mu_1, \dots, \mu_m) + \partial Y/\partial X_1(X_1-\mu_1) + \partial Y/\partial X_2(X_2-\mu_2) + \dots + \partial Y/\partial X_m(X_m-\mu_m), \tag{2}$$

where  $\partial Y/\partial X_j$  is the partial derivative of  $Y$  with respect to  $X_j$  calculated at  $(\mu_1, \dots, \mu_m)$ .

The variance is thus approximated by:

$$VAR(Y)=\sigma^2 \cong (\partial Y/\partial X_1)^2 \cdot \sigma_1^2 + \dots + (\partial Y/\partial X_m)^2 \cdot \sigma_m^2, \tag{3}$$

whereas the expected value is:

$$E(Y) = \mu = Y(\mu_1, \dots, \mu_m). \tag{4}$$

Obviously, in order to evaluate the results of a particular model in terms of  $\mu$  and  $\sigma$  an explicit formula (1) must be constructed and partial derivatives with respect to all variables (2) must be calculated. The procedure is quite simple in itself and does not require specific knowledge of the probability distributions of the variables involved besides the expected value and standard deviations. However, in cases of complex models having many variables the derivation and calculation of the partial derivatives may become elaborate.

**3. Modeling by intervals**

Recently it has been suggested to use intervals in order to represent uncertainties in connection with worst - and best - case (WBC) evaluation of economic consequences of technological development, Schjær-Jacobsen [17,18]. The interval approach was originally developed in 1962 by Moore [14,15] in order to be able to keep track of the lower and upper bounds to the exact result when carrying out numeri-

cal calculations on digital computers with a finite number of significant digits. Following Moore [15] we define an interval number as an ordered pair  $[a;b]$  of real numbers with  $a \leq b$ . It may also be defined as an ordinary set of real numbers  $x$  such that  $a \leq x \leq b$ , or:

$$[a;b] = \{x \mid a \leq x \leq b\}. \quad (5)$$

If the basic operations addition, subtraction, multiplication, and division is denoted by the symbol  $\#$  we can define operations on two intervals  $I_1 = [a_1;b_1]$  and  $I_2 = [a_2;b_2]$  based on the set-theoretic formulation:

$$I_1 \# I_2 = \{x \# y \mid a_1 \leq x \leq b_1, a_2 \leq y \leq b_2\}. \quad (6)$$

Instead of this set-theoretic definition we may give alternative definitions in terms of endpoints of the resulting intervals by the formulas quoted in Table 1. It should be mentioned that whereas the rules for basic calculations with intervals are commutative and associative they are not distributive.

In the case where the interval function to be evaluated is a monotone function of the intervals over their entire range the calculation is trivial. The resulting interval end points may simply be calculated at the end points of the variables. In the case of the interval function not being monotonic it is a different situation.

A simple example is the calculation of the non-monotonic expression  $Y = I(1 - I)$  where  $I$  is an interval,  $I = [0;1]$ . Straight forward application of the above mentioned formulas gives the result  $Y = [0;1]$  which obviously is a too wide interval. According to the fundamental definition of basic operations on intervals based upon set-theory (6), Moore [15], the narrowest possible resulting interval should be  $Y = [0;1/4]$ . In this paper the terms “true” or “correct” is used to indicate the narrowest possible bounds that can be calculated for an uncertain number. By using iterative global optimization methods, see Hansen [6], correct results may be obtained to an accuracy specified by the user. This feature has been implemented in the add-in module Interval Solver 2000 for MS-Excel, Hyvönen and De Pascale [8,9].

#### 4. Modeling by fuzzy numbers

Since the introduction by Zadeh [22] fuzzy sets and fuzzy numbers have found a wide range of applica-

tions within the areas of engineering, management, and finance. A fuzzy set is a class of objects with a continuum of grades of membership defined by a membership function ranging from zero to one. The fuzzy set concept provides a convenient way of keeping precisely track with imprecise, vague, and uncertain informative statements such as “the class of all large investments”, “costs will be considerably reduced in the coming period”, and “the turn over will be a little larger next year”.

Following Zadeh [22] a fuzzy set  $A$  in  $X$  where  $X$  is a space of points (objects) with a generic element of  $X$  denoted by  $x$ , i.e.  $X = \{x\}$ , is characterized by a membership function  $f_A(x)$  which associates with each point in  $X$  a real number in the interval  $[0;1]$ . The value of the membership function  $f_A(x)$  at  $x$  represents the “grade of membership” of  $x$  in  $A$ . Thus the closer the value of  $f_A(x)$  to unity, the higher the grade of membership of  $x$  in  $A$ . Note, that when  $A$  is an ordinary set, i.e. non-fuzzy, the membership function can take only two values 0 and 1.

In other words, a fuzzy set is a set of ordered pairs  $(x, f_A(x))$  as:

$$A = \{(x, f_A(x)) \mid x \in X\}. \quad (7)$$

It may also be useful to define the ordinary (non-fuzzy) set  $A_\alpha$  as the  $\alpha$ -cut of  $A$ :

$$A_\alpha = \{x \in X \mid f_A(x) \geq \alpha, 0 \leq \alpha \leq 1\}. \quad (8)$$

In this paper we are mainly interested in the concept of fuzzy numbers as a means of representing uncertain or fuzzy information, Dubois and Prade [3,4]. In addition to the simplest fuzzy number, namely the interval (5), we also make use of the triangular fuzzy number, Chiu and Park [2], that can be defined the following way using real numbers  $a \leq c \leq b$ :

$$f(x) = (x-a)/(c-a) \quad a \leq x \leq c, \quad (9a)$$

$$f(x) = (b-x)/(b-c) \quad c \leq x \leq b, \quad (9b)$$

$$f(x) = 0 \quad \text{otherwise.} \quad (9c)$$

Full basic operations on triangular fuzzy numbers may be facilitated by introducing the left  $L(\alpha)$  and right  $R(\alpha)$  representation of a fuzzy number  $F$ , refer to the  $\alpha$ -cut (8):

$$F = [L(\alpha); R(\alpha)], \quad (10a)$$

where

$$L(\alpha) = a + (c-a)\alpha \quad \alpha \in [0,1], \quad (10b)$$

and

$$R(\alpha) = b + (c-b)\alpha \quad \alpha \in [0,1]. \quad (10c)$$

Observe that in (10),  $F$  is written as an interval with upper and lower bounds depending on  $\alpha$ . This means that addition, subtraction, multiplication, and division can be carried out by means of the interval formulas in Table 1 by all values of  $\alpha$ . In the general case membership functions of arbitrary complexity may result and make the practical calculations prohibitively elaborate. One way of overcoming this difficulty is to limit the calculations to a small number of values of  $\alpha$ . As an example of this consider the triple estimate defined by  $\alpha$ -cuts corresponding to the values  $\alpha = 0$  and  $\alpha = 1$ , refer to (8).

Based on the above and also as a generalization of Kaufmann and Gupta [10], we may now define basic operations on triple estimate triangular fuzzy numbers  $F_1$  and  $F_2$  by the formulas found in Table 1:

$$F_1 = [a_1; c_1; b_1],$$

$$F_2 = [a_2; c_2; b_2].$$

In the general case when calculating uncertain functions with triple estimate fuzzy number arguments care must be taken in order to produce true lower and upper limits. As an example consider the non-monotonic expression  $Y = F(1 - F)$ , where  $F$  denotes a triple estimate,  $F = [0; 1/2; 1]$ . The true triple estimate value of this expression turns out to be  $Y = [0; 1/4; 1/4]$ , which is not obtained by straightforward calculations because the variable  $F$  appears twice in the expression. The remedy may be to use Interval Solver 2000, Hyvönen and De Pascale [8,9], to calculate true lower and upper bounds by application of global optimization.

## 5. Discounted cash flow analysis

Considered the net present value NPV calculated over  $n$  periods by the following function:

$$NPV = a_0 + a_1(1+r_1)^{-1} + a_2(1+r_1)^{-1}(1+r_2)^{-1} + \dots + a_n(1+r_1)^{-1}(1+r_2)^{-1} \dots (1+r_n)^{-1}, \quad (11)$$

where  $r_1, r_2, \dots, r_n$  are the discount rates of interest and the net cash flow in the  $i^{\text{th}}$  period is given by the following expression:

$$a_i = X_{i1}X_{i2} + X_{i3} + X_{i4} + \dots + X_{im} \quad i=0, \dots, n. \quad (12)$$

In practice, investment decisions are often taken based upon the NPV being positive and in cases of uncertain input parameters it is important to know the resulting uncertainty of NPV in order to match with the risk preferences of the decision makers, Schjær-Jacobsen *et al.* [19].

## 6. Stochastic modeling

Let the relevant uncertain parameters in (11) and (12) be represented by the following known independent stochastic variables:

$$X_{ij} = \{\mu_{ij}, \sigma_{ij}\} \quad i = 0, \dots, n \text{ and } j = 1, \dots, m,$$

and

$$r_i = \{\mu_i, \sigma_i\} \quad i = 0, \dots, n. \quad (13)$$

While the expected value  $\mu$  of NPV is easily calculated by inserting the expected values of the stochastic variables in formulas (12) and (13), we get for the variance  $\sigma^2$  of NPV by means of (3):

$$\begin{aligned} \sigma^2 \cong & (\partial NPV / \partial X_{01})^2 \sigma_{01}^2 + (\partial NPV / \partial X_{02})^2 \sigma_{02}^2 \\ & + \dots + (\partial NPV / \partial X_{0m})^2 \sigma_{0m}^2 \\ & + (\partial NPV / \partial X_{11})^2 \sigma_{11}^2 + (\partial NPV / \partial X_{12})^2 \sigma_{12}^2 \\ & + \dots + (\partial NPV / \partial X_{1m})^2 \sigma_{1m}^2 \\ & + (\partial NPV / \partial X_{21})^2 \sigma_{21}^2 + (\partial NPV / \partial X_{22})^2 \sigma_{22}^2 \\ & + \dots + (\partial NPV / \partial X_{2m})^2 \sigma_{2m}^2 + \dots \\ & + (\partial NPV / \partial X_{n1})^2 \sigma_{n1}^2 + (\partial NPV / \partial X_{n2})^2 \sigma_{n2}^2 \\ & + \dots + (\partial NPV / \partial X_{nm})^2 \sigma_{nm}^2 \\ & + (\partial NPV / \partial r_1)^2 \sigma_1^2 + (\partial NPV / \partial r_2)^2 \sigma_2^2 \\ & + \dots + (\partial NPV / \partial r_n)^2 \sigma_n^2, \end{aligned} \quad (14)$$

where  $\partial NPV / \partial X_{ij}$  is the partial derivative of NPV with respect to  $X_{ij}$  calculated at  $\mu_{ij}$  and  $\partial NPV / \partial r_i$  is the partial derivative of NPV with respect to  $r_i$  calculated at  $\mu_i$ .

For the partial derivatives with respect to the  $X$ 's in (14) we get for the 0<sup>th</sup> period:

$$\begin{aligned}\partial NPV / \partial X_{01} &= X_{02}, \\ \partial NPV / \partial X_{02} &= X_{01}, \\ &\vdots \\ \partial NPV / \partial X_{0j} &= 1 \quad j = 3, \dots, m.\end{aligned}\quad (15)$$

For the first period we get

$$\begin{aligned}\partial NPV / \partial X_{11} &= X_{12} (1 + r_1)^{-1}, \\ \partial NPV / \partial X_{12} &= X_{11} (1 + r_1)^{-1}, \\ &\vdots \\ \partial NPV / \partial X_{1j} &= (1 + r_1)^{-1} \quad j = 3, \dots, m.\end{aligned}\quad (16)$$

Likewise, for the second period we get:

$$\begin{aligned}\partial NPV / \partial X_{21} &= X_{22} (1 + r_1)^{-1} (1 + r_2)^{-1}, \\ \partial NPV / \partial X_{22} &= X_{21} (1 + r_1)^{-1} (1 + r_2)^{-1}, \\ &\vdots \\ \partial NPV / \partial X_{2j} &= (1 + r_1)^{-1} (1 + r_2)^{-1} \quad j=3, \dots, m.\end{aligned}\quad (17)$$

For the  $i^{\text{th}}$  period, the partial derivatives are:

$$\begin{aligned}\partial NPV / \partial X_{i1} &= X_{i2} (1 + r_1)^{-1} (1 + r_2)^{-1} \dots (1 + r_i)^{-1}, \\ \partial NPV / \partial X_{i2} &= X_{i1} (1 + r_1)^{-1} (1 + r_2)^{-1} \dots (1 + r_i)^{-1}, \\ \partial NPV / \partial X_{ij} &= (1 + r_1)^{-1} (1 + r_2)^{-1} \dots (1 + r_i)^{-1}, \\ &j = 3, \dots, m.\end{aligned}\quad (18)$$

Finally, the partial derivatives with respect to the rate of interests in (14) are calculated.

$$\begin{aligned}\partial NPV / \partial r_1 &= -a_1 (1 + r_1)^{-2} - a_2 (1 + r_1)^{-2} (1 + r_2)^{-1} \\ &\quad - \dots - a_n (1 + r_1)^{-2} (1 + r_2)^{-1} \dots (1 + r_n)^{-1}, \\ \partial NPV / \partial r_2 &= -a_2 (1 + r_1)^{-1} (1 + r_2)^{-2} - \dots \\ &\quad - a_n (1 + r_1)^{-1} (1 + r_2)^{-2} \dots (1 + r_n)^{-1}, \\ &\quad \vdots \\ \partial NPV / \partial r_n &= -a_n (1 + r_1)^{-1} (1 + r_2)^{-1} \dots (1 + r_n)^{-2}.\end{aligned}\quad (19)$$

## 7. Comparison of alternative modeling methods

Consider the case of a possible investment in developing, manufacturing, and selling of an industrial product over a period of 5 years modeled by the formulas (11) to (19). Prior to the investment decision being taken a discounted cash flow analysis must be carried out in order to analyze the consequences of the future cash flows being known only with uncertainty. The following numerical calculations are all concerned with the same case, however with alternative representations of the uncertain variables.

We start out by representing all the uncertain values by means of their minimum and maximum values and the results of this interval modeling are shown in Table 2a.

The resulting net present value is [-3.532; 3.514] which is probably not satisfying a decision maker to authorize this project from a financial point of view because of a substantial possibility of creating a negative net present value.

The next method to be considered is the modeling by stochastic variables and the input variables in Table 2b have been generated from uniform probability distributions corresponding to the intervals in Table 2a.

All input variables in Table 2b have been created from the input variables  $[a; b]$  in Table 2a by transforming them into uniform probability distributions with  $\mu = (a+b)/2$  and  $\sigma^2 = (b-a)^2/12$ .

The resulting net present value is seen to have an expected value of -111 and a standard deviation of 470. A risk averse decision maker would probably not go ahead with the project because of the negative expected net present value and the relatively low probability of a positive result.

**Table 1.** Formulas for basic calculations with alternative representations of uncertain variables.

	<b>Independent stochastic variables</b> $\{\mu; \sigma\} = \{\mu_1; \sigma_1\} \# \{\mu_2; \sigma_2\}$	<b>Intervals</b> $[a; b] = [a_1; b_1] \# [a_2; b_2]$	<b>Triple estimates</b> $[a; c; b] = [a_1; c_1; b_1] \# [a_2; c_2; b_2]$
<b>Addition</b>	$\mu = \mu_1 + \mu_2;$ $\sigma^2 = \sigma_1^2 + \sigma_2^2$	$a = a_1 + a_2;$ $b = b_1 + b_2$	$a = a_1 + a_2;$ $c = c_1 + c_2;$ $b = b_1 + b_2$
<b>Subtraction</b>	$\mu = \mu_1 - \mu_2;$ $\sigma^2 = \sigma_1^2 + \sigma_2^2$	$a = a_1 - b_2;$ $b = b_1 - a_2$	$a = a_1 - b_2;$ $c = c_1 - c_2;$ $b = b_1 - a_2$
<b>Multiplication</b>	$\mu = \mu_1 \cdot \mu_2;$ $\sigma^2 \cong \sigma_1^2 \cdot \mu_2^2 + \sigma_2^2 \cdot \mu_1^2$	$a = \min(a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2);$ $b = \max(a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2)$	$a = \min(a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2);$ $c = c_1 c_2;$ $b = \max(a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2)$
<b>Division</b>	$\mu = \mu_1 / \mu_2;$ $\sigma^2 \cong \sigma_1^2 / \mu_2^2 + \sigma_2^2 \cdot \mu_1^2 / \mu_2^4,$ if $\mu_2 \neq 0$	$a = \min(a_1 / b_2, a_1 / a_2, b_1 / b_2, b_1 / a_2);$ $b = \max(a_1 / b_2, a_1 / a_2, b_1 / b_2, b_1 / a_2),$ if $0 \notin [a_2; b_2]$	$a = \min(a_1 / b_2, a_1 / a_2, b_1 / b_2, b_1 / a_2);$ $c = c_1 / c_2;$ $b = \max(a_1 / b_2, a_1 / a_2, b_1 / b_2, b_1 / a_2),$ if $0 \notin [a_2; b_2]$

**Table 2a.** Discounted cash flow analysis by interval analysis (Interval Solver 2000, overall absolute and relative precision  $10^{-6}$ ). Input variables are shaded.

(\$1000)	YEAR 0	YEAR 1	YEAR 2	YEAR 3	YEAR 4
<b>Turnover</b>		[4.200;5.200]	[12.400;14.100]	[15.900;18.100]	[13.800;15.600]
<b>Margin (%)</b>		[44,50;45,50]%	[45,00;47,00]%	[45,50;48,50]%	[44,00;48,00]%
<b>Direct cost</b>		[-2.886;-2.289]	[-7.755;-6.572]	[-9.865;-8.188]	[-8.736;-7.176]
<b>Margin</b>		[-1.869;2.366]	[5.580;6.672]	[7.234;8.779]	[6.072;7.488]
<b>Marketing cost</b>	[-1.050;-950]	[-1.000;-800]	[-975;-700]	[-800;-600]	[-800;-600]
<b>Indirect production cost</b>		[-950;-700]	[-1.375;-1.225]	[-675;-525]	[-675;-525]
<b>RD&amp;E cost</b>	[-3.050;-2.950]	[-1.700;-1.400]	[-350;-250]	[-150;-50]	[-150;-50]
<b>Operating income</b>	[-4.100;-3.900]	[-1.781;-534]	[2.880;4.452]	[5.609;7.604]	[4.447;6.313]
<b>Investment</b>	[-5.100;-4.900]	[-2.200;-1.900]			[0;700]
<b>Net cash flow</b>	[-9.200;-8.800]	[-3.981;-2.434]	[2.880;4.452]	[5.609;7.604]	[4.447;7.013]
<b>Rate of interest r (%)</b>		[8,50;9,50] %	[9,00;11,00] %	[9,50;12,50] %	[10,50;13,50] %
<b>Discounted cash flow</b>	[-9.200;-8.800]	[-3.669;-2.223]	[2.369;3.764]	[4.102;5.871]	[2.865;4.901]
<b>Net present value</b>	[-3.532;3.514]				

**Table 2b.** Discounted cash flow analysis by stochastic variables, formulas (11) to (19). Input variables (shaded cells) are derived from uniform probability distributions corresponding to the interval input variables in Table 2a, however converted to the form  $\{\mu; \sigma\}$ .

(\$1000)	YEAR 0	YEAR 1	YEAR 2	YEAR 3	YEAR 4
<b>Turnover</b>		[4.700;289]	[13.250;491]	[17.000;635]	[14.700;520]
<b>Margin (%)</b>		[45,00;0,29]%	[46,00;0,58]%	[47,00;0,87]%	[46,00;1,15]%
<b>Direct cost</b>		[-2.585;160]	[-7.155;276]	[-9.010;368]	[-7.938;328]
<b>Margin</b>		[2.115;131]	[6.095;239]	[7.990;333]	[6.762;293]
<b>Marketing cost</b>	[-1.000;29]	[-900;58]	[-838;79]	[-700;58]	[-700;58]
<b>Indirect production cost</b>		[-825;72]	[-1.300;43]	[-600;43]	[-600;43]
<b>RD&amp;E cost</b>	[-3.000;29]	[-1.550;87]	[-300;29]	[-100;29]	[-100;29]
<b>Operating income</b>	[-4.000;41]	[-1.160;182]	[3.657;257]	[6.590;342]	[5.362;303]
<b>Investment</b>	[-5.000;58]	[-2.050;87]			[350;202]
<b>Net cash flow</b>	[-9.000;71]	[-3.210;202]	[3.657;257]	[6.590;342]	[5.712;364]
<b>Rate of interest r (%)</b>		[9,00;0,29]%	[10,00;0,58]%	[11,00;0,87]%	[12,00;0,87]%
<b>Discounted cash flow</b>	[-9.000;71]	[-2.945;184]	[3.050;215]	[4.952;262]	[3.832;251]
<b>Net present value</b>	[-111; 470]				

**Table 3a.** Discounted cash flow analysis by triple estimates. Minimum and maximum values identical to Table 2a, most possible values obtained by ordinary (crisp) calculations. Input variables in shaded cells.

(\$1000)	YEAR 0	YEAR 1	YEAR 2	YEAR 3	YEAR 4
<b>Turnover</b>		[4.200;5.000;5.200]	[12.400;14.000;14.100]	[15.900;18.000;18.100]	[13.800;15.500;15.600]
<b>Margin (%)</b>		[44,50;45,00;45,50]%	[45,00;46,00;47,00]%	[45,50;47,00;48,50]%	[44,00;46,00;48,00]%
<b>Direct cost</b>		[-2.886;-2.750;-2.289]	[-7.755;-7.560;-6.572]	[-9.865;-9.540;-8.188]	[-8.736;-8370;-7.176]
<b>Margin</b>		[-1.869;2.250;2.366]	[5.580;6.440;6.672]	[7.234;8.460;8.779]	[6.072;7.130;7.488]
<b>Marketing cost</b>	[-1.050;-1.000;-950]	[-1.000;-900;-800]	[-975;-800;-700]	[-800;-700;-600]	[-800;-700;-600]
<b>Ind. prod. cost</b>		[-950;-750;-700]	[-1.375;-1.275;-1.225]	[-675;-575;-525]	[-675;-575;-525]
<b>RD&amp;E cost</b>	[-3.050;-3.000;-2.950]	[-1.700;-1.500;-1.400]	[-350;-300;-250]	[-150;-100;-50]	[-150;-100;-50]
<b>Oper. income</b>	[-4.100;-4.000;-3.900]	[-1.781;-900;-534]	[2.880;4.065;4.452]	[5.609;7.085;7.604]	[4.447;5.755;6.313]
<b>Investment</b>	[-5.100;-5.000;-4.900]	[-2.200;-2.000;-1.900]			[0;600;700]
<b>Net cash flow</b>	[-9.200;-9.000;-8.800]	[-3.981;-2.900;-2.434]	[2.880;4.065;4.452]	[5.609;7.085;7.604]	[4.447;6.355;7.013]
<b>Rate of interest</b>		[8,50;9,00;9,50]%	[9,00;10,00;11,00]%	[9,50;11,00;12,50]%	[10,50;12,00;13,50]%
<b>Disco. cash flow</b>	[-9.200;-9.000;-8.800]	[-3.669;-2.661;-2.223]	[2.369;3.390;3.764]	[4.102;5.324;5.871]	[2.865;4.263;4.901]
<b>Net present val.</b>	[-3.532;1.317;3.514]				

**Table 3b.** Discounted cash flow analysis by stochastic variables, formulas (11) - (19). Input variables (shaded cells) are derived from triangular probability distributions corresponding to the triple estimate input variables in Table 3a, however converted to the form  $\{\mu; \sigma\}$ .

(\$1000)	YEAR 0	YEAR 1	YEAR 2	YEAR 3	YEAR 4
<b>Turnover</b>		[4.800;216]	[13.500;389]	[17.333; 507]	[14.800;374]
<b>Margin (%)</b>		[45,00;0,20] %	[46,00;0,41] %	[47,00; 0,61] %	[46,00;0,82] %
<b>Direct cost</b>		[-2.640;119]	[-7.290;217]	[-9.186; 289]	[-7.992;236]
<b>Margin</b>		[2.160;98]	[6.210;187]	[8.147; 261]	[6.808;211]
<b>Marketing cost</b>	[-1.000;20]	[-900;41]	[-825;57]	[-700; 41]	[-700;41]
<b>Ind. prod. cost</b>		[-800;54]	[-1.292;31]	[-592; 31]	[-592;31]
<b>RD&amp;E cost</b>	[-3.000;20]	[-1.533;62]	[-300;20]	[-100; 20]	[-100;20]
<b>Oper. income</b>	[-4.000;28]	[-1.073;134]	[3.793;199]	[6.755; 266]	[5.416;218]
<b>Investment</b>	[-5.000;41]	[-2.033;62]			[433;155]
<b>Net cash flow</b>	[-9.000;50]	[-3.106;148]	[3.793;199]	[6.755; 266]	[5.849;267]
<b>Rate of interest</b>		[9,00;0,20] %	[10,00;0,41] %	[11,00; 0,61] %	[12,00;0,61] %
<b>Disco. cash flow</b>	[-9.000;50]	[-2.850;134]	[3.163;167]	[5.075; 203]	[3.924;182]
<b>Net present val.</b>	[313;355]				

The Tables 3a and 3b display similar results of calculations, however starting out in Table 3a with triple estimates created by inserting out most possible values between the minimum and maximum values shown in Table 2a.

Obviously, from Table 3a, the net present value is now [-3.532; 1.317; 3.514] and thus the investment appears to be more attractive than it was previously (Table 2a) although it is still quite possible to end up with a negative net present value. Note, that taking only the conventional “crisp” net present value of 1.317 into account would leave the decision maker with no reservations toward to the profitability of the project. Results using stochastic variables in Table 3b show an expected net present value of 313 and a standard deviation of 355. It is worth noting that all input variables in Table 3b have been created from the input variables  $[a;c;b]$  in Table 3a by transforming them into triangular probability distributions with  $\mu = (a+b+c)/3$  and  $\sigma^2 = (a^2+b^2+c^2-ab-ac-bc)/18$ .

An interesting observation based on the calculations reported in Tables 3a and 3b follows. Although the input variables (triangular fuzzy numbers in Table 3a and triangular probability distributions in Table 3b) have been derived from identical basic data, i.e. triple estimates  $[a;c;b]$ , the resulting net present values may lead to alternative rhetorical arguments.

In the case of triangular fuzzy number representation the most possible net present value is 1.317 whereas the expected net present value in case of a representation by triangular probability distributions is 313, a quite substantial difference. Does this difference make the project less attractive from a probability point of view than from a possibility point of view? Not necessarily, the two figures may not be directly compared! The point is that although the basic input data are identical the arithmetic operations in the probability case are different from the operations in the possibility case and thus generates different numerical values of the resulting net present values.

In the case of probabilities one may say that the characteristic value of the net present value (i.e. the expected value) is determined solely by the expected values of the input distributions.

In the case of possibilities the characteristic value of the net present value (i.e. the most possible value) is determined solely by the most possible value of the input distributions. In other words, the most probable values of the probabilistic input variables are not propagated through the calculations defined by stochastic arithmetic (see the formulas in Table 1).

## 8. Conclusion

In this paper alternative ways of modeling economic uncertainty have been investigated. Economic variables have been represented and handled computationally in the following ways:

- Ordinary numbers, also called “crisp” numbers, computed by a standard spreadsheet program MS-Excel.
- Double estimates, like intervals  $[a;b]$  computed by Interval Solver 2000 as an add-in module for MS-Excel, and stochastic variables  $\{\mu;\sigma\}$  computed by means of approximate formulas developed for the particular economic models considered.
- Triple estimates of the form  $[a;c;b]$ , being simplifications of triangular fuzzy numbers, computed by ordinary calculations in combination with Interval Solver 2000.

One may ask which one of the modeling techniques mentioned should be preferred. First of all, it should be remembered that we try to handle imperfect knowledge by representing it in terms of uncertain numbers. Consequently, that representation should be chosen that most closely reflects the kind of imperfect knowledge at hand. Or on the contrary, that kind of knowledge should be retrieved that most closely enables us to make conclusions relevant to the decision situation at hand.

In the case of the interval modeling approach, clearly you only need to know the true minimum and maximum values of the input variables. By applying Interval Solver 2000 you will then easily arrive at the true minimum and maximum values of the output variables. The worst and best case argument goes like this: provided the input variables stay within their bounds the output variables also will in the case of the triple estimate modeling approach you might even benefit from the advantage of being able to identify the most possible outcome by tracing the most possible values of the input variables. Once an ordinary spread sheet model for “crisp” calculations is developed it may be intervallized by Interval Solver 2000. From a communicative point of view the triple estimate approach is easily understood as an extension of an ordinary “crisp” calculation by adding true worst and best cases.

On the other hand, if the independent variables are known in terms of expected values and standard deviations only the stochastic approach might be useful.

The complication is that for each particular model the formulas governing the resulting standard deviations have to be derived. However, once derived the application is straight forward. Specific knowledge about the precise shape of the probability distributions is not needed and will not be known for the dependent variables either, except for the fact that in case of more complex models (typically more than 15-20 variables, according to experience) the dependent variables will be close to normal distributions. Thus 1% or 5% fractiles may be used instead of worst- and best cases to indicate practical limits to probable outcomes. One of the drawbacks of the stochastic model still is the difficulty of communicating with people unfamiliar with probabilities and statistics. Further to the approaches in this paper comparisons with Monte Carlo simulations may be found in Schjær-Jacobsen [20]. In this paper we have focused on the representation and calculation aspects in order to evaluate the modeling characteristics and qualities of competing approaches. Nevertheless, it should be born in mind that the crucial point in practical applications to decision making still is the ability of the decision maker to “know” something about the future states of the world subject to uncertainty and then, simultaneously, to be able to handle that uncertainty in adequate ways.

## References

- [1] Bayes, T., 1763, *An Essay Towards Solving a Problem in the Doctrine of Chances*. Reprinted 1970 in Pearson and Kendall (editors), *Studies in the History of Statistics and Probability*.
- [2] Chiu, C. U. and Park, C. S., 1994, Fuzzy cash flow analysis using present worth criterion. *The Engineering Economist*, 39(2), 113-138.
- [3] Dubois, D. and Prade, H., 1978, Operations on fuzzy numbers. *International Journal of System Science*, 9, 613-626.
- [4] Dubois, D. and Prade, H., 1979, Fuzzy real algebra: Some results. *Fuzzy Sets and Systems*, 2, 327-348.
- [5] Dubois, D. and Prade, H., 1988, *Possibility Theory - An Approach to Computerized Processing of Uncertainty*. Plenum Press, New York.
- [6] Hansen, E., 1992, *Global Optimization Using Interval Analysis*. Marcel Dekker, New York, USA.
- [7] Hertz, D. B., 1964, Risk analysis in capital investment. *Harvard Business Review*, January/February, 95-106.
- [8] Hyvönen, E. and De Pascale, S., 1999, *A New Basis for Spreadsheet Computing: Interval Solver for Microsoft Excel*. Proceedings of the 11<sup>th</sup> Innovative Applications of Artificial Intelligence, AAAI Press, Menlo Park, California.
- [9] Hyvönen, E. and De Pascale, S., 2000, *Interval Solver 2000 for Microsoft Excel*. User's Guide, Version 4.0. Program Release 4.0.0.2. Delisoft Ltd., Helsinki, Finland.
- [10] Kaufmann, A. and Gupta, M. M., 1988, *Fuzzy Mathematical Models in Engineering and Management Science*. Elsevier Science Publishers.
- [11] Knight, F. H., 1921, *Risk, Uncertainty and Profit*. Houghton Mifflin, Boston.
- [12] Kyläheiko, K., 1995, *Coping with Technology: A Study on Economic Methodology and Strategic Management of Technology*. Doctoral dissertation, Lappeenranta University of Technology, Finland.
- [13] Laplace, P. S., 1795, *A Philosophical Essay on Probabilities*. Dover, New York, USA.
- [14] Moore, R. E., 1962, *Interval Arithmetic and Automatic Error Analysis in Digital Computing*. Ph.D. dissertation, Stanford University.
- [15] Moore, R. E., 1966, *Interval Analysis*. Prentice-Hall, Englewood Cliffs, New Jersey, USA.
- [16] Ramsey, F. P., 1926, *Truth and Probability*. Chapter 7 of 1931, Braithwaite (editor), *The Foundations of Mathematics and Other Logical Essays*, Routledge, London, UK.
- [17] Schjær-Jacobsen, H., 1996, A new method for evaluating worst-and best-case (WBC) economic consequences of technological development. *International Journal of Production Economics*, 46-47, 241-250.
- [18] Schjær-Jacobsen, H., 1997, *Handling Economic Risks and Uncertainties of Production Technology Investments*. Proceedings of the 14<sup>th</sup> International Conference on Production Research, Osaka, Japan, 1, 346-349.
- [19] Schjær-Jacobsen, H., Thuneby, J. and Madsen, K., 2000, Representation, calculation, and decision making under economic uncertainty. *Ledelse og Erhvervsøkonomi*, 64(3), 133-148.
- [20] Schjær-Jacobsen, H., 2004, Modeling of economic uncertainty. *Fuzzy Economic Review*, IX(2), 49-73.
- [21] Von Neumann, J. and Morgenstern, O., 1944, *Theory of Games and Economic Behaviour*. Princeton University Press, Princeton, New Jersey, USA.
- [22] Zadeh, L. A., 1965, Fuzzy sets. *Information and Control*, 8, 338-353.